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Computation of Feasible and Invariant Sets for Interpolation-based MPC

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Abstract: The terminal invariant set plays a key role in the stabilizing MPC (Model Predictive Control) formulation. When control gains of the terminal local control laws and corresponding feasible and invariant sets are given, the existing interpolation methods unite them to enlarge the stabilizable region and enhance performance. In this paper, when an invariant set is given, an algorithm is proposed to find another invariant set such that their convex hull is maximized and also invariant. Numerical examples show that the set of the stabilizable initial state of the MPC is enlarged by the terminal constraint set computed by an interpolation-based approach.

Keywords: Interpolation, invariant set, model predictive control, terminal set.

1. INTRODUCTION

Model predictive control (MPC) is a type of controller which minimizes a cost index over a finite horizon in the receding horizon manner. When the state is measured at a sampling time, an optimization problem is solved, and the first part of the optimal solution is applied to the system. Then, the whole procedure is repeated at the next sampling time. The advantage of the MPC over conventional controllers is the ability to handle the state and control constraints [1].

The concept of the terminal penalty is widely used to guarantee the stability of the MPC both for linear systems, e.g., [2,3], and nonlinear systems, e.g., [4–6]. The terminal penalty function is an upper bound of the infinite horizon cost needed to drive the state trajectory to the origin when the initial condition is in the terminal region. In order to have closed-loop stability, an artificial stability constraint is used to force the terminal state to belong to the terminal region defined by a level set of the terminal penalty function. In view of this, it is quite important to have a terminal region as large as possible since a large terminal set can have a large set of the stabilizable initial state. Various approaches to find possibly larger terminal sets are available in the literature, e.g., [7–10]. On the other hand, several interpolation strategies have been proposed to enlarge the invariant set [11,12]. The study in [13] formulates an interpolation method based on the decomposition of the measured state. As a result, it requires a large degree of freedom. Alternative approaches have considered linear interpolation for the whole trajectory, for example, methods in [12] and [14]. In [14], the proposed algorithm uses only one optimization variable. The survey by [15] formulates a time-varying terminal cost for MPC by interpolating several terminal costs. Reference [16] describes the use of the interpolation method for a continuous-time system.

The existing studies focus on how to combine given invariant sets. On the other hand, less attention has been paid to find larger invariant sets for the interpolation. Since the interpolation of the given invariant sets leads to their convex hull, and the convex hull is again feasible and invariant, it is essential to have large invariant sets, called basic sets, for the interpolation. With this in mind, this paper proposes a systematic way to find the basic sets. In other words, when a feasible and invariant set is given, the proposed method computes another feasible and invariant set such that the convex hull of the two sets becomes as large as possible. The proposed algorithm uses linear algebra and linear matrix inequalities (LMIs).

The paper is constructed as follows: Section 2 describes preliminaries and the problem definitions. Section 3 presents the proposed method to find the basic sets for an interpolation-based approach. Based on the proposed method, it is presented that an interpolation-based approach can enlarge the terminal invariant set of a linear MPC at the expense of adding one more scalar variable to the on-line optimization problem for the MPC. The interpolation method is discussed in Section 4. The proposed algorithms are validated through numerical simulations, which are given in Section 5.

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Notation: Given any sets E_1 and $E_2 \subset \mathbb{R}^n$, the union of these sets is denoted by $E_1 \cup E_2$ and the convex hull of $E_1 \cup E_2$ is $co\{E_1 \cup E_2\}$. For a square matrix A, $A > 0$ denotes a positive definite matrix, $A < 0$ a negative definite matrix, and $det(A)$ the determinant of matrix A. Frobenius norm of matrix A is represented by $\|A\|_F$. Denote matrix I as an identity matrix with the appropriate dimension. A vector in $\mathbb{R}^n$ whose elements are all zero is denoted by $\mathbf{0}_n$. Given a geometry $\triangle$ in n-dimensional space with $n \geq 2$, $V(\triangle)$ is the area of $\triangle$ if $n = 2$ and it is the volume of $\triangle$ if $n > 2$.

2. PRELIMINARIES AND PROBLEM SETUP

Consider a constrained linear time-invariant discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

$$|u(k)| \leq u_{max}, \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is state, $u(k) \in \mathbb{R}^m$ the input, u_{max} the input constraint, and the inequality in (2) denotes element-wise inequality.

Lemma 1 [21]**:** Suppose that system (1) is stabilizable, then there exist matrices $X = X^\top > 0$, $Z = Z^\top$ such that

$$\begin{bmatrix} X & (AX+BY)^\top \\ AX+BY & X \end{bmatrix} > 0, \tag{3a}$$

$$\begin{bmatrix} Z & Y \\ Y^\top & X \end{bmatrix} > 0, \tag{3b}$$

$$Z_{jj} \leq u_{max}^2, \quad j = 1, \cdots, m. \tag{3c}$$

Also, assume that the initial condition $x(0)$ belongs to the set $E = \{x \in \mathbb{R}^n | x^\top X^{-1} x \leq 1\}$. Then, the control law $u(k) = Kx(k)$, with $K = YX^{-1}$, guarantees the stability of closed loop system (1) and satisfies the input constraint (2). Besides, the state variable $x(k)$ belongs to the set E for all $k \geq 0$.

Note that the LMI conditions (3) is a sufficient condition guaranteeing that the set E is feasible and invariant. The interpolation method is established by considering more than one control laws $u(k) = K_i x(k)$, $i = 1, \ldots, N$, $N > 1$. Define matrices $X_i > 0$, such that E_i is an ellipsoid centered at the origin

$$E_i = \{x \in \mathbb{R}^n | x^\top X_i^{-1} x \leq 1\}, \quad i \in \{1, \cdots, N\}, \tag{4}$$

and is the corresponding feasible and invariant set for K_i. For example, by considering two feasible and invariant sets E_1 and E_2, the resulting feasible and invariant set constructed by the interpolation method is the convex hull of $E_1 \cup E_2$ [1,3,5,13–15,17]. The following lemma presents an example of the interpolation method.

Lemma 2 [14]**:** For system (1), let ellipsoids $E_i \subset \mathbb{R}^n$, $i = 1, 2, \ldots, N$ be feasible and invariant under the corresponding state feedback gain K_i. Define the convex combination of K_i for all $i = 1, 2, \ldots, N$ as

$$K_{int} := \left(\sum_{i=1}^N \gamma_i Y_i \right) \left(\sum_{i=1}^N \gamma_i X_i \right)^{-1} =: Y_{int} X_{int}^{-1},$$

where $Y_i = K_i X_i$, $\gamma_i \in [0, 1]$, and $\sum_{i=1}^N \gamma_i = 1$. Then, there exists γ_i guaranteeing the stability the closed-loop system (1) under control law

$$u(k) = K_{int} x(k). \tag{5}$$

Furthermore, the control law (5) makes the following become a feasible and invariant set

$$E_{int} = \{x \in \mathbb{R}^n | x^\top X_{int}^{-1} x \leq 1\}.$$

Note that the union of E_{int} for all possible $\gamma_i \in [0, 1]$ satisfying $\sum_{i=1}^N \gamma_i = 1$ is the convex hull of $E_1, \ldots, E_N$. Thus, a possible larger feasible and invariant set can be obtained by selecting $E_1, \ldots, E_N$ such that $E_i \not\subseteq E_j, j \neq i, i, j \in [1, N]$. This paper is directed to design a systematic way to find the sets $E_1, \ldots, E_N$ resulting in as large as possible $co\{E_1 \cup \ldots \cup E_N\}$.

For the sake of simplicity, only two feasible and invariant sets, i.e., E_1 and E_2, are taken into account in the rest of this paper. In order to enlarge the feasible and invariant set by the interpolation method, E_1 and E_2 have to be chosen such that the convex hull of $E_1 \cup E_2$ is as large as possible. Such ellipsoids are called the basic sets for the interpolation method in this paper. For this purpose, it is assumed that a feasible and invariant set of the closed-loop system (1) is given by an ellipsoidal set. Given an ellipsoid $E \subset \mathbb{R}^n$ centered at the origin with $X > 0$, the following lemmas describe the properties of E.

Lemma 3 [18]**:** Principal axis of an ellipsoid is given by eigenvectors of X, with radii $\sqrt{\lambda_i}$ where λ_i, $i = 1, \ldots, n$ are the eigenvalues of X.

Lemma 4 [19]**:** Consider two ellipsoids $E_1 \subset \mathbb{R}^n$ and $E_2 \subset \mathbb{R}^n$ centered at the origin. Then, $E_1 \subseteq E_2 \Leftrightarrow X_1 \leq X_2$.

Lemma 5 [18]**:** Consider an ellipsoid $E \subset \mathbb{R}^2$ with $X > 0$. Rotating E by the angle of θ about the origin gives a new ellipsoid E' with $X' = R(\theta)^\top X R(\theta) > 0$ where $R(\theta)$ denotes the rotation matrix defined by

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{6}$$

2.1. Basic sets for interpolation

In this section, for a given feasible and invariant set E_1, a systematic method is proposed to find another feasible and invariant set such that the convex hull of both sets becomes as large as possible. To this end, consider an ellipsoidal set E_d obtained by rotating E_1. For a two-dimensional ellipsoid, the rotated ellipsoid is defined in Lemma 5. Meanwhile, the rotation matrix for the n-dimensional case is developed based on the plane formed by any two coordinated axes [20]. Denote the chosen axes as x_a and x_b. The rotation matrix $R^{ab}(\theta) \in \mathbb{R}^{n\times n}$ for the axis x_a in the direction of x_b by the angle θ is defined as follows:

$$R^{ab}(\theta) = \left\{ r_{ij} \left| \begin{array}{lcll} r_{ii} & = & 1, & i \neq a, j \neq b \\ r_{aa} & = & \cos\theta, & \\ r_{bb} & = & \cos\theta, & \\ r_{ab} & = & -\sin\theta, & \\ r_{ba} & = & \sin\theta, & \\ r_{ij} & = & 0, & \text{otherwise.} \end{array} \right. \right.$$

Based on the rotation matrix $R^{ab}(\theta)$, define a positive definite matrix X_d as $X_d := R^{ab}(\theta)^\top X_1 R^{ab}(\theta)$ and its corresponding ellipsoid E_d. Note that if the chosen axis x_a and x_b have different length and $0^o < \theta < 180^o$, then $E_1 \not\subseteq E_d$ and $E_d \not\subseteq E_1$. Furthermore, if E_d is a feasible and invariant set, then the interpolation method can be applied using E_1 and E_d. However, E_d is not necessarily feasible and invariant. Hence, it cannot be used as the basic set of interpolation. In this paper, the rotated ellipsoid E_d is considered as a reference set to construct the second invariant set E_2 in order to make the convex hull of $E_1 \cup E_2$ as large as possible.

For a given feasible and invariant set E_1 and its rotated ellipsoid E_d, X_1^{-1} can be decomposed by $X_1^{-1} = L^\top L$ as it is a real symmetric and positive definite matrix. Denote W and D as matrices corresponding to the eigenvector and eigenvalue of $I - L^\top X_d L$, respectively. The following optimization problem is considered to find the second feasible and invariant set E_2 using the feasible and invariant set E_1 and ellipsoid E_d.

$$\min_{T,Y_2,Z_2} \|X_2 - X_d\|_F \tag{7a}$$

$$\text{subject to} \quad X_2 = L^{-1}[I - WTW^\top]L^{\top^{-1}},$$
$$T = diag([t_i]), \quad D = diag([d_i]),$$

$$\begin{bmatrix} X_2 & (AX_2 + BY_2)^\top \\ AX_2 + BY_2 & X_2 \end{bmatrix} > 0, \tag{7b}$$

$$\begin{bmatrix} Z_2 & Y_2 \\ Y_2^\top & X_2 \end{bmatrix} > 0, \; X_2 > 0, \tag{7c}$$

$$Z_{2,jj} \leq u_{max}, \quad j = 1, \ldots, n \tag{7d}$$

$$\text{for } i = 1, \ldots, n,$$
$$t_i < 0, \;\; i \in \{i | d_i < 0\}, \tag{7e}$$
$$0 < t_i < 1, \;\; i \in \{i | d_i \geq 0\}.$$

Denote X_2^{ij} and X_d^{ij} as the element in the ith row and jth column of X_2 and X_d, respectively. Note that Frobenius norm $\|.\|_F$ has the following definition

$$\|X_2 - X_d\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |X_2^{ij} - X_d^{ij}|}.$$

In view of this, the optimization problem (7) aims to find another invariant set E_2 such that it is as similar as possible to E_d. The following theorem shows that the resulting E_2 from optimization (7) is also feasible and invariant.

Theorem 1: Suppose that a feasible and invariant ellipsoidal set E_1 with $X_1 > 0$ is given. If the problem (7) is feasible, then the resulting set E_2 is a feasible and invariant set of the closed-loop system (1) with $u(k) = K_2 x(k)$ and $K_2 = Y_2 X_2^{-1}$. Furthermore, if the chosen axis x_a and x_b have different length and $0^o < \theta < 180^o$, then both $E_2 \not\subseteq E_1$ and $E_2 \not\supseteq E_1$ hold true.

Note that both $E_2 \not\subseteq E_1$ and $E_2 \not\supseteq E_1$ provide that not all the elements of E_1 belong to E_2, and the other way around.

Proof: If the optimization problem (7) is feasible, all the constraints are satisfied. The constraint (7b) implies invariance, and (7c) and (7d) mean feasibility.

Consider a coordinate change to transform E_1 into a ball $E_1' \subset \mathbb{R}^n$

$$E_1' = \{y \in \mathbb{R}^n | y^\top y \leq 1\}.$$

Note that the matrix representation of ball E_1' is an identity matrix $I \in \mathbb{R}^{n\times n}$. Implementing the same coordinate transformation to the ellipsoid E_2 and E_d yields E_2' and E_d', respectively

$$E_2' = \{y \in \mathbb{R}^n | y^\top X_2'^{-1} y \leq 1\},$$
$$E_d' = \{y \in \mathbb{R}^n | y^\top X_d'^{-1} y \leq 1\}.$$

Then, $X_2' = L X_2 L^\top$ and $X_d' = L X_d L^\top$. Due to this transformation, $X_2 \leq X_1 \Leftrightarrow X_2' \leq I$ and $X_2 \geq X_1 \Leftrightarrow X_2' \geq I$. Thus, both $E_2 \not\subseteq E_1$ and $E_2 \not\supseteq E_1$ can be proved by showing $E_2' \not\subseteq E_1'$ and $E_2' \not\supseteq E_1'$.

As the chosen axis x_a and x_b have different length and $0^o < \theta < 180^o$, it follows that $E_d \not\subseteq E_1$ and $E_d \not\supseteq E_1$, or equivalently $E_d' \not\subseteq E_1'$ and $E_d' \not\supseteq E_1'$. Consequently, $I - L X_d L^\top = WDW^{-1}$ is indefinite. As a result, there exist at least one positive and at least one negative d_i constructing the diagonal matrix D. In view of (7e), there exist at least one $t_i < 0$ and at least one $0 < t_i < 1$.

Since $X_2 = L^{-1}[I - WTW^\top]L^{\top^{-1}}$, X_2' can be decomposed by

$$X_2' = I - WTW^\top. \tag{8}$$

In order to have both $E_2' \not\subseteq E_1'$ and $E_2' \not\supseteq E_1'$ hold true, $I - X_2'$ has to be an indefinite matrix. Note that $I - X_2' = WTW^\top$ has at least one positive and at least one negative eigenvalue as T is indefinite. Hence, both $E_2' \not\subseteq E_1'$ and $E_2' \not\supseteq E_1'$ hold true. □

Using the optimization problem (7), the procedure to find the basic sets for interpolation generating convex hull is given by 1) finding a feasible and invariant set E_1 using any kind of methods in the literature, for instance, the largest volume of E_1 can be obtained by maximizing the trace of matrix X_1 [21], 2) by changing θ gradually, computing candidate ellipsoids for E_2 based on (7) such that the convex hull of $E_1 \cup E_2$ is as large as possible. This procedure is less conservative than feedback gain tuning because the choice of scalar θ is bounded by $[0^o, 180^o]$. From Theorem 1, note that if the problem is feasible, the chosen axis x_a and x_b have different length, and the parameter θ is in between 0^o and 180^o, then both $E_2 \not\subseteq E_1$ and $E_2 \not\supseteq E_1$ hold true. Furthermore, the objective function $||X_2 - X_d||_F$ is equivalent to

$$\min_{G,E} \ \text{trace}(G)$$

subject to

$$\begin{bmatrix} G & (X_2 - X_d)^\top \\ X_2 - X_d & I \end{bmatrix} \geq 0. \tag{9}$$

In the light of Theorem 1, E_2 can be obtained by changing θ gradually. Suppose that two invariant sets E_1 and E_2 are considered for the interpolation method. Then, it is better to have an efficient method to choose θ such that it makes the convex hull of $E_1 \cup E_2$ as large as possible. Such a problem can be solved by comparing the area or volume of the convex hull. However, measuring the volume of the convex hull even for a two-dimensional case is computationally hard. Hence, the focus is placed on the lower bound of the volume of $co\{E_1 \cup E_2\}$.

Theorem 2: Given two ellipses $E_1 \subset \mathbb{R}^n$ and $E_2 \subset \mathbb{R}^n$ such that the following holds true

$$E_2 \not\subseteq E_1, \ E_2 \not\supseteq E_1. \tag{10}$$

Consider matrix $L \in \mathbb{R}^{n\times n}$ satisfying $X_1^{-1} = L^\top L$ and denote ρ_i as the ith eigenvalue of $L^\top X_2 L$. The lower bound of the area of $co\{E_1 \cup E_2\}$ is given by

$$2^n |\det(L)| \cdot V(\triangle^n),$$

where $\triangle^n$ denotes the convex hull of the following vertices

$$\mathbf{0}_n, s^1, s^2, \ldots, s^n, \tag{11}$$

and s^i is the ith column of matrix $S = diag([\upsilon_i]) \in \mathbb{R}^{n\times n}$ with $\upsilon_i = max(1, \sqrt{\rho_i})$.

Proof: Note that $\triangle^n$ is an n-dimensional polytope, or widely known as n-simplex. At first, to show the idea of the proof, a two-dimensional case is considered. Then, it is shown that the proof can be generalized for the n-dimensional case.

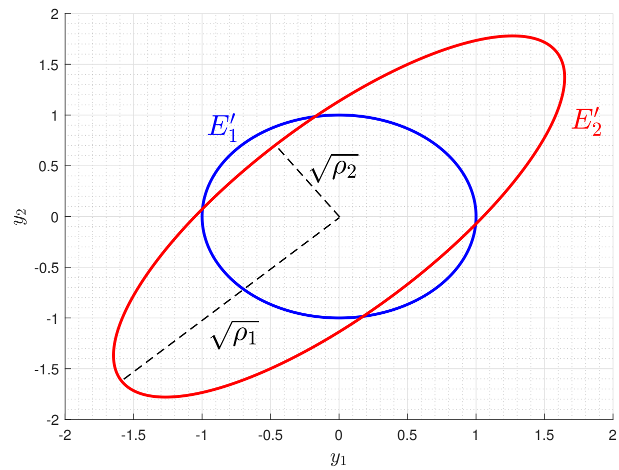


Fig. 1. Ellipsoid E_2' and circle E_1'.

Consider two ellipsoids $E_1 \subset \mathbb{R}^2$ and $E_2 \subset \mathbb{R}^2$. By implementing the same coordinate transformation as in the proof of Theorem 1 to both sets, a circle with radius one $E_1' = \{y \in \mathbb{R}^2 | y^\top y \leq 1\}$ and an ellipse $E_2' = \{y \in \mathbb{R}^2 | y^\top X_2' y \leq 1\}$ are obtained. The matrix representation of E_2' is given by $X_2' = LX_2L^\top$. Note that $X_2 \leq X_1 \Leftrightarrow X_2' \leq I$ and $X_2 \geq X_1 \Leftrightarrow X_2' \geq I$. In the light of (10), both $X_2 \not\leq X_1$ and $X_2 \not\geq X_1$ hold true. As a result, it follows that

$$X_2' \not\leq I, \ X_2' \not\geq I, \tag{12}$$

which means that both $E_2' \not\subseteq E_1'$ and $E_2' \not\supseteq E_1'$ are true. Denote matrix $\Sigma = diag([\rho_i])$ consisting of the eigenvalues of X_2'. In view of (12), $I - X_2'$ is indefinite, which means then there exist at one $\rho_i > 1$ and one $0 < \rho_i < 1$. Fig. 1 illustrates E_1' and E_2'. Without loss of generality, it is assumed that the length of the major axis of E_2' corresponds to the first eigenvalue of X_2'.

Due to the transformation, the area of the convex hull of $E_1 \cup E_2$ is equal to the determinant of the linear transformation times the area of the convex hull of $E_1' \cup E_2'$. Namely, it follows that

$$\text{area of } co\{E_1 \cup E_2\} = |\det(L)| \cdot \text{area of } co\{E_1' \cup E_2'\}.$$

Furthermore, there exists a rotation matrix $R(\theta)$ to rotate the ellipse E_2' such that the axes of the rotated ellipse is parallel to the coordinate, which is illustrated in Fig. 2. Denote E_2'' as the rotated ellipse and X_2'' as the matrix representation of E_2''. Note that X_2'' has the same eigenvalues as those of X_2'. With this in mind, since the circle is invariant under rotation, the area of $E_1' \cup E_2''$ is equivalent with the area of $E_1' \cup E_2'$. It follows that the area of convex hull of $E_1' \cup E_2'$ and that of $E_1' \cup E_2''$ are equal. In light of this, the lower bound of the area of $co\{E_1' \cup E_2''\}$ is also a lower bound of the area of $co\{E_1' \cup E_2'\}$.

Fig. 3 illustrates the set E_1' and E_2'' with additional rhombus set $\mathcal{R}$. Note that the area of rhombus $\mathcal{R}$ can

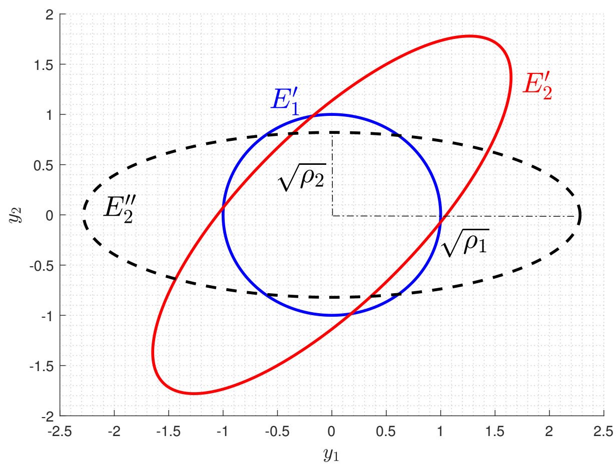


Fig. 2. Rotating E_2' yields ellipsoid E_2''.

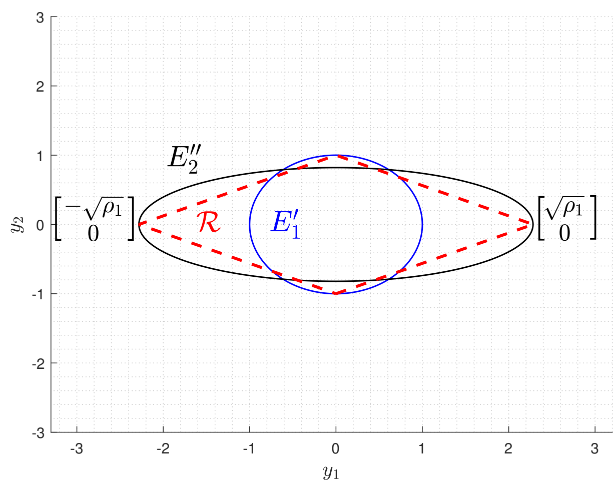


Fig. 3. Set rhombus $\mathcal{R}$ can be seen as a lower bound of the area of $co\{E_1' \cup E_2''\}$.

be seen as a lower bound of the area of $co\{E_1' \cup E_2''\}$ as $\mathcal{R} \subset co\{E_1' \cup E_2''\}$. Furthermore, the area of rhombus $\mathcal{R}$ is equivalent to four times the area of a triangle defined by the vertices:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{\rho_1} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{13}$$

Note that these vertices correspond to (11) with one $\rho_i > 1$ and one $0 < \rho_i < 1$. It is well-known that a triangle is a simplex with dimension one, which is denoted by $\triangle^2$. Hence, the lower bound of area of $co\{E_1 \cup E_2\}$ is given by $4|\det(L)|$ times the area of the simplex $\triangle^2$ spanned by (13). Note that $[\sqrt{\rho_1}; 0]$ is the intersection of $co\{E_1' \cup E_2''\}$ and the positive y_1-axis. Similarly, $[0; 1]$ is the intersection of $co\{E_1' \cup E_2''\}$ and the positive y_2-axis.

On the ground of this result, the proof can be generalized for the n-dimensional case. Consider ellipsoids $E_1 \subset \mathbb{R}^n$ and $E_2 \subset \mathbb{R}^n$ with $n > 2$ such that (10) is satisfied. Implementing the same coordinate transformation yields a sphere $E_1' \subset \mathbb{R}^n$ and an ellipsoid $E_2' \subset \mathbb{R}^n$. Then, the following holds true

$$\begin{aligned} &\text{volume of } co\{E_1 \cup E_2\} \\ &= |\det(L)| \cdot \text{volume of } co\{E_1' \cup E_2'\}. \end{aligned}$$

Furthermore, by applying the rotation of axes to the ellipsoid E_2', the axis-aligned ellipsoid E_2'' is obtained. Note that the eigenvectors of X_2'' are orthogonal as X_2' is a real symmetric matrix. As a result, it only requires one rotation of the axis to obtain E_2''. Similarly to the one described for the two-dimensional case, the volume of the resulting $co\{E_1' \cup E_2'\}$ is equivalent to that of $co\{E_1' \cup E_2''\}$. Then, we are ready to investigate the lower bound of the volume of $co\{E_1' \cup E_2''\}$. To do that, $co\{E_1' \cup E_2''\}$ is divided into subspaces according to the sign of each variable, which results in 2^n equal orthants. This is possible as E_2'' is an axis-aligned ellipsoid and $co\{E_1' \cup E_2''\}$ is centered at the origin. For example, $co\{E_1' \cup E_2''\}$ in the three dimensional space can be divided into eight equal octant domains. As a result, the volume of $co\{E_1' \cup E_2''\} \subset \mathbb{R}^n$ is 2^n times the volume of the geometry contained in any orthant. Denote $\mho \subset \mathbb{R}^n$ as the subset of $co\{E_1' \cup E_2''\}$ in the non-negative orthant. Then, the volume of $co\{E_1' \cup E_2''\}$ is 2^n times the volume of $\mho$. With this in mind, the lower bound of volume of $co\{E_1' \cup E_2''\}$ can be computed by looking into the lower bound of the volume of $\mho$. In the n-dimensional case, denote $s^1, s^2, \ldots, s^n \in \mathbb{R}^n$ as the vertices corresponding to the intersection of $\mho$ with the $y_1, y_2, \ldots, y_n$-axis, respectively. Then, s_i is a vector with all zero elements except for the ith element. The non-zero element in the vertex s_i is given by $\upsilon_i = max(1, \sqrt{\rho_i})$. Recall that $\sqrt{\rho_i}$ is the length of the ith principal axis of ellipsoid X_2''. Note that the largest $\triangle^n$ contained in $\mho$ is the convex hull of $\mathbf{0}_n$ and the vertices $s^1, s^2, \ldots, s^n$. For instance, in the three dimensional case, suppose that the length of the principal axes of E_2'' which are parallel to the y_1- and y_2-axis are larger than one, then $\triangle^3$ is a tetrahedron defined by

$$\mathbf{0}_3, \begin{bmatrix} \sqrt{\rho_1} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{\rho_2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since $\triangle^n \subset \mho$, then the volume of $\triangle^n$ can be a lower bound of $\mho$. As a result, the lower bound of $co\{E_1' \cup E_2''\}$ is given by

$$2^n \cdot V(\triangle^n). \qquad \square$$

Theorem 2 shows that a rhombus (n-simplex) can be used to estimate the lower bound of the area (volume) of the convex hull area made by two ellipses (ellipsoids). The lower bound of convex hull of rectangles is also studied in [24]. In the light of Theorem 2, the ellipsoid E_2 can be chosen such that the corresponding n-simplex has the largest volume. The volume of the n-simplex defined by

Algorithm 1:

1: **input:** Feasible invariant ellipsoid set E_1, chosen rotation axis x_a and x_b
2: **for** $\theta = \{\bar{\theta}, \underline{\theta}\}$ **do**
3: Solve **P1:**

$$\min_{G,T,Y_2,Z_2} \ \mathrm{trace}(G)$$

subject to (7b), (7c), (7d), (7e), (9)

4: **if** feasible **then** Collect $\Omega : \{(X_2^j, det(S^j)\}$, $S^j = diag(\upsilon_i), \upsilon_i = max(1, \sqrt{1-t_i}), T = diag([t_i])$
5: **end if**
6: **end for**
7: Choose $X_2 = X_2^j$ such that $max(det(S^j))$. Basic set: E_1 and E_2.

(11) is given by $\frac{1}{n!}\det(S)$ [22]. In this paper, given the invariant set E_1, the set E_2 is computed by solving (7) and comparing the volume of n-simplex representing the lower bound of $co\{E_1 \cup E_2\}$. The proposed algorithm is formally written in Algorithm 1.

Remark 1: By observing (8), $\rho_i = 1 - t_i$ is derived. In Algorithm 1, the sets generated by solving optimization problem **P1** for different θ can be seen as the candidates for E_2. Then, among these sets, E_2 is chosen such that the corresponding matrix S has the largest determinant. In view of Theorem 2, this means that E_2 is selected such that a lower bound of the volume of $co\{E_1 \cup E_2\}$ is the largest. Note that the sets obtained **P1** are invariant. Thus, Algorithm 1 can still be used to attain more than two basic sets for an interpolation-based approach. In order to find N basic sets, the set E_1 and θ have be chosen such that **P1** is feasible at least N times in a selected range $\theta = \{\bar{\theta}, \underline{\theta}\}$.

3. ENLARGING THE TERMINAL INVARIANT SET IN A LINEAR MPC

The purpose of this section is to show that an interpolation method can be applied to enlarge the terminal set of a linear MPC.

Consider discrete-time LTI linear systems:

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \tag{14}$$

where (A, B) is stabilizable. The system is subject to the control input constraint where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$. The system is subject to the control input constraint

$$u \in \mathbb{U} := \{u | -u_{max} \le u \le u_{max}\}, \tag{15}$$

where $u_{max} > 0$ is constant. It is assumed that the state $x(k)$ is measurable.

The optimization problem for the finite horizon MPC is formulated as follows:

$$\begin{aligned} &\min_{\mathcal{U}} J_N(\mathcal{U}, x(k)) \\ &\text{subject to} \\ &x(k+i+1|k) = Ax(k+i|k), +Bu(k+i|k), \quad i \in [0, N-1], \\ &x(k|k) = x(k), \\ &u(k+i|k) \in \mathbb{U}, \ x(k+N|k) \in \mathcal{X}_f, \end{aligned} \tag{16}$$

where

$$\begin{aligned} &J_N(\mathcal{U}, x(k)) \\ &= \sum_{i=0}^{N-1} l(x(k+i|k), u(k+i|k)) + V(x(k+N|k)), \\ &l(x,u) = x^\top Qx + u^\top Ru, \end{aligned}$$

and $\mathcal{U} = [u(k|k), u(k+1|k), \ldots, u(k+N-1|k)]$ denotes the control prediction over the prediction horizon and $x(k)$ is the current state measurement. The matrix $Q \ge 0$ and $R > 0$ are the weighting matrices for the state and input variables, respectively. Set $\mathcal{X}_f$ denotes the terminal penalty set.

The procedure of the linear MPC is started by solving the optimization problem when state $x(k)$ is measured at the kth sampling time. As a result, the optimal sequence $\mathcal{U}^*(k)$ is obtained. Then, $u^*(k) = [\mathbb{I}_m \ \mathbf{0}]\mathcal{U}^*$ is applied to the plant where m denotes the size of input vector u. Afterward, the same procedure is repeated at the next sampling time.

The following lemma describes stabilizing ingredients such that closed-loop stability is guaranteed.

Lemma 6 [1]**:** Suppose that the optimization problem (16) is initially feasible and that the following holds.

- $\exists K \in \mathbb{R}^{m\times n}$, $Kx \in \mathbb{U}$, $\forall x \in \mathcal{X}_f$,
- $(A+BK)x \in \mathcal{X}_f$, $\forall x \in \mathcal{X}_f$,
- $V(x) - V((A+BK)x) \ge x^\top (Q + K^\top RK)x$, $\forall x \in \mathcal{X}_f$.

Then, recursive feasibility is guaranteed and the closed-loop system is asymptotically stable.

In this paper, $\mathcal{X}_f$ and $V(x)$ are chosen as follows:

$$\mathcal{X}_f = \{x \in \mathbb{R}^n | x^\top X^{-1} x \le \alpha\}, \tag{17a}$$

$$V(x) = x^\top X^{-1} x, \tag{17b}$$

where $X > 0$. Suppose that there exist $Y \in \mathbb{R}^{m\times n}$, $X \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{m\times m}$, and $\alpha \in \mathbb{R}$ that satisfy the following linear matrix inequalities (LMIs):

$$\begin{bmatrix} X & (AX+BY)^\top & (Q^{\frac{1}{2}}X)^\top & (R^{\frac{1}{2}}Y)^\top \\ AX+BY & X & 0 & 0 \\ Q^{\frac{1}{2}}X & 0 & \alpha & 0 \\ R^{\frac{1}{2}}Y & 0 & 0 & \alpha \end{bmatrix} \ge 0, \tag{18a}$$

$$\begin{bmatrix} Z & Y \\ Y^\top & X \end{bmatrix} \geq 0, \tag{18b}$$

$$Z \leq diag(u_{max}^2). \tag{18c}$$

Then, Lemma 6 holds true for (17) and $K = YX^{-1}$. See [17] for the derivations. In this paper, let us consider two sets of $(K_1, X_1, Z_1, \alpha_1)$ and $(K_2, X_2, Z_2, \alpha_2)$ satisfying (18). As a result, pairs of $(\mathcal{X}_{f,1}, X_1)$ and $(\mathcal{X}_{f,2}, X_2)$ can be used as the stabilizing ingredients of the terminal set and cost for the MPC given by (16) where $\mathcal{X}_{f,i} := \{x \in \mathbb{R}^n | x^\top(k+N|k)X_i^{-1}x(k+N|k) \leq \alpha_i\}$, $i = 1, 2$. Having this, consider the following optimization problem for the interpolation-based MPC.

$$\min_{\mathcal{U},\gamma} \quad J_N(\mathcal{U}, x(k)) \tag{19}$$

subject to

$$x(0) = x(k),$$
$$x(k+i+1|k) = Ax(k+i|k) + Bu(k+i|k), \quad i \in [0, N-1],$$
$$u(k+i|k) \in \mathbb{U}, \; x(k+N|k) \in \mathcal{X}_{f,int},$$
$$\gamma(k) \in [0, 1],$$

where

$$J_N(\mathcal{U}, x(k)) = \sum_{i=0}^{N-1} l(x(k+i|k), u(k+i|k)) + V(x(k+N|k)),$$
$$X_{int}(\gamma(k)) = \gamma(k)X_1 + (1-\gamma(k))X_2,$$
$$V(x) = x^\top X_{int}^{-1} x,$$
$$\mathcal{X}_{f,int} = \{x \in \mathbb{R}^n | x^\top X_{int}^{-1}(\gamma(k))x \leq \gamma(k)\alpha_1 + (1-\gamma(k))\alpha_2\}.$$

Note that, compared with usual MPC formulation, γ is added to the optimization variable. Moreover, $\mathcal{X}_{f,int} \in co\{\mathcal{X}_{f,1} \cup \mathcal{X}_{f,2}\}$. In this optimization problem, γ is used to interpolate two matrices X_1 and X_2. Thus, the final state prediction at each step $x(k+N|k)$ is required to belong to the convex hull of $\mathcal{X}_{f,1}$ and $\mathcal{X}_{f,2}$.

The following theorem shows how the interpolation method leads to an enlarged terminal invariant set, which means that the region of attraction of the MPC is enlarged as well. The result is parallel to the result in [15].

Theorem 3: For the input-constrained stabilizable system (14), suppose that $(K_1, X_1, Z_1, \alpha_1)$ and $(K_2, X_2, Z_2, \alpha_2)$ satisfy (18), and that the optimization problem (19) for the interpolation-based MPC is initially feasible. Then, recursive feasibility and closed stability are guaranteed.

Proof: Since (18) holds true for $(K_1, X_1, Z_1, \alpha_1)$ and $(K_2, X_2, Z_2, \alpha_2)$, the following holds true for any $\gamma \in [0, 1]$

$$\gamma \begin{bmatrix} Z_1 & Y_1 \\ Y_1^\top & X_1 \end{bmatrix} + (1-\gamma) \begin{bmatrix} Z_2 & Y_2 \\ Y_2^\top & X_2 \end{bmatrix} = \begin{bmatrix} Z_{int} & Y_{int} \\ Y_{int}^\top & X_{int} \end{bmatrix} \geq 0,$$

$$\gamma Z_1 + (1-\gamma)Z_2 = Z_{int} \leq diag(u_{max}^2),$$

where $X_{int} = \gamma X_1 + (1-\gamma)X_2$, $Y_{int} = \gamma Y_1 + (1-\gamma)Y_2$, and $Z_{int} = Z_1 + (1-\gamma)Z_2$. The same technique can be done to (18a) to show that

$$\begin{bmatrix} X_{int} & A_{int}^\top & (Q^{\frac{1}{2}}X_{int})^\top & (R^{\frac{1}{2}}Y_{int})^\top \\ A_{int} & X_{int} & 0 & 0 \\ Q^{\frac{1}{2}}X_{int} & 0 & \alpha_{int} & 0 \\ R^{\frac{1}{2}}Y_{int} & 0 & 0 & \alpha_{int} \end{bmatrix} \geq 0.$$

where $A_{int} = AX_{int} + BY_{int}$ and $\alpha_{int} = \gamma\alpha_1 + (1-\gamma)\alpha_2$. Having this, $K_{int} = Y_{int}X_{int}^{-1}$, $V(x) = x^\top X_{int}^{-1}x$ and $\mathcal{X}_{f,int} = \{x \in \mathbb{R}^n | x^\top X_{int}^{-1} x \leq \alpha_{int}\}$ satisfy Lemma 6 for $\gamma \in [0, 1]$, meaning that recursive feasibility and asymptotically stability are guaranteed under the assumption that (19) is initially feasible. See Appendix for the rest of the proof. □

Remark 2: It is important to note that γ is a part of the optimization variables in the MPC problem (19), which is a difference compared with usual MPC schemes. This means that the time-varying terminal cost $V(x)$ and terminal invariant set $\mathcal{X}_{f,int}$ are constructed at each time step k. In view of this, the resulting optimal $\gamma^*(k)$ gives a trade-off between performance index (smaller $V(x)$) and feasibility (larger $\mathcal{X}_{f,int}$).

Remark 3: Note that the optimization problem for the MPC given in (19) is closely affiliated with the work in [15]. However, in this paper, the method described in the previous section is used to find $\mathcal{X}_{f,i}$, $i = 1, 2$, such that as large as possible $co\{\mathcal{X}_{f,1} \cup \mathcal{X}_{f,2}\}$ can be obtained. To be more precise, given a terminal set $\mathcal{X}_{f,1}$, $\mathcal{X}_{f,2}$ is generated by ensuring that the pair (K_2, X_2) obtained from Algorithm 1 satisfies LMIs (18). Furthermore, given Lemma 2, the MPC problem (19) can be generalized to the case where more than two feasible and invariant sets are considered. However, the interpolation method presented in this section can only be used when the system is linear. When the system is nonlinear, the interpolation method discussed in [23] can be employed to enlarge the terminal set of the nonlinear MPC. Nevertheless, the proposed method discussed in Section 3 can still be used to find the basic sets for [23] with some minor modifications. Possible future research includes solving the tracking MPC problem using an interpolation-based approach.

In the next section, the proposed method is investigated through a numerical simulation.

4. EXAMPLES

In this section, the effectiveness of the proposed method to find the basic sets for the interpolation-based MPC is demonstrated using a numerical example. Consider a linear system (1) with

$$A = \begin{bmatrix} 0.385 & 0.33 \\ 0.21 & 0.59 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

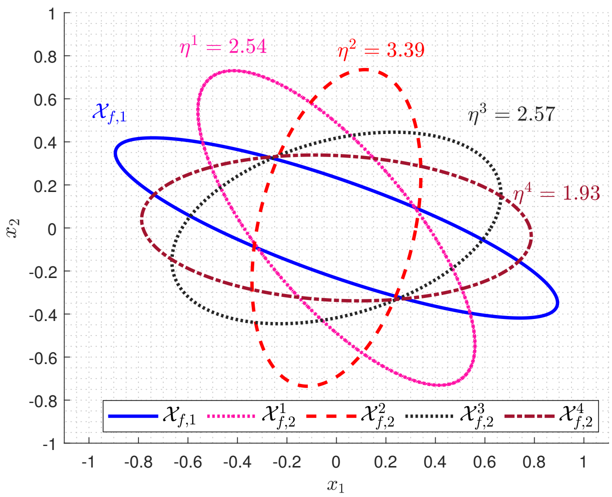
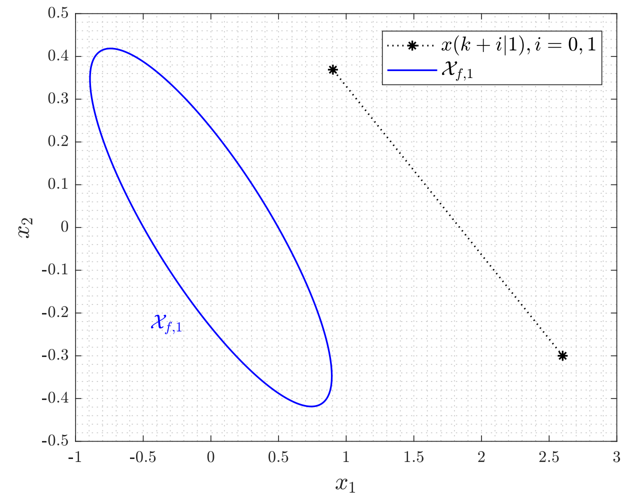


Fig. 4. Candidate sets for E_2: E_2^1, E_2^2, E_2^3, and E_2^4. Denote η^j as the determinant of S^j. The set E_2^3 is chosen as E_2 as η^3 is the largest.

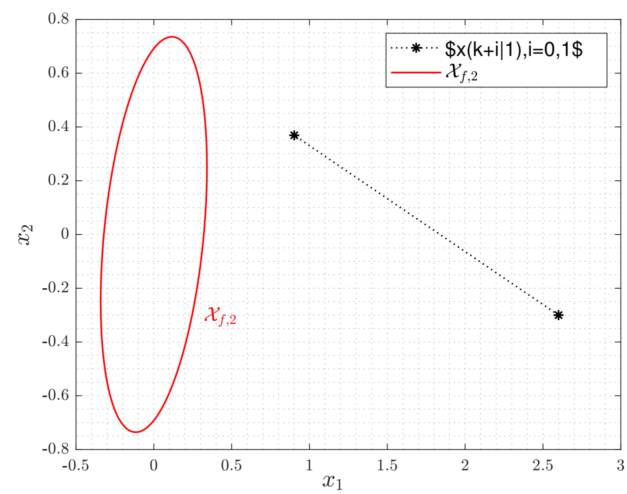
and $u_{max} = 0.5$. A known algorithm of maximizing the trace of X_1 is used to obtain the initial set $\mathcal{X}_{f,1}$. The state feedback gain K_1 is computed by applying pole placement method with desired pole $0.1 \pm 0.06j$. Given $Q = diag([1,1])$ and $R = 1$, the resulting $\mathcal{X}_{f,1}$ is represented by matrix

$$X_1 = \begin{bmatrix} 0.798 & -0.310 \\ -0.310 & 0.175 \end{bmatrix}.$$

Having $\mathcal{X}_{f,1}$, Algorithm 1 is employed with four different values of θ. Fig. 4 shows resulting feasible and invariant sets $\mathcal{X}_{f,2}^1$, $\mathcal{X}_{f,2}^2$, $\mathcal{X}_{f,2}^3$, and $\mathcal{X}_{f,2}^4$ which are used as the resulting candidates of $\mathcal{X}_{f,2}$. The set E_2 is selected among E_2^1, E_2^2, E_2^3, and E_2^4. Denote η^j as the determinant of S^j. The set $\mathcal{X}_{f,1}^2$ is selected as the set $\mathcal{X}_{f,2}$ because it has the largest η. In this case, it is easy to see that $\mathcal{X}_{f,2}^2$ and $\mathcal{X}_{f,1}$ yields the largest convex hull compared to $co\{\mathcal{X}_{f,1} \cup \mathcal{X}_{f,2}^j\}$, with $j = 1, 3, 4$. After the basic sets E_1 and E_2 for interpolation are selected, an interpolation-based MPC is calculated by solving optimization problem (19) in order to enlarge the region of attraction of the MPC. The conventional MPC is computed with $N = 1$ and terminal set $\mathcal{X}_{f,i}$, $i = 1, 2$ as the terminal sets. Given initial state $x_0 = [2.6, 0.3]$, Fig. 5 shows that the predicted state does not belong to the desired terminal set, meaning that the MPC is initially infeasible. However, when the interpolation-based MPC is applied, the terminal set can be enlarged by employing the convex hull of the given two sets as can be seen in Fig. 6. This means that the region of attraction of the MPC is also enlarged by having the convex hull of the given two sets as the terminal set. Fig. 7 depicts the stability of the origin, which means that Theorem 3 is verified. Moreover, Fig. 8 shows the stability of the origin from several different initial conditions.



(a) Terminal set $\mathcal{X}_{f,1}$.



(b) Terminal set $\mathcal{X}_{f,2}$.

Fig. 5. For given $N = 1$ and $[2.6, 0.3]$, the terminal state does not belong to the terminal set $\mathcal{X}_{f,i}$, $i = 1, 2$, i.e., the optimization problem for the MPC is initially infeasible.

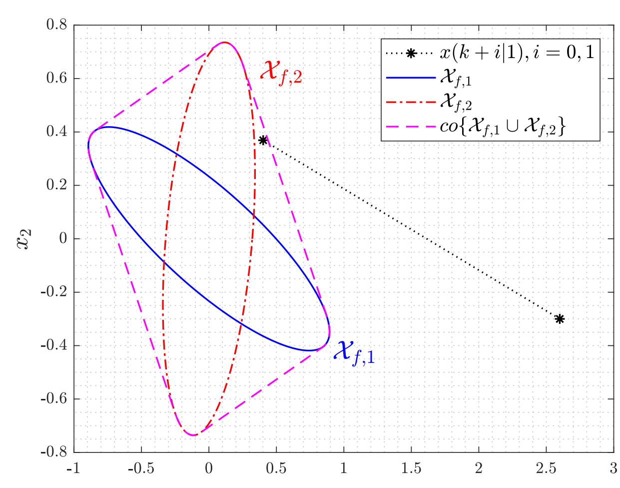


Fig. 6. Predicted state at time $k = 1$ of the MPC with interpolation method belongs to the terminal set which is $co\{\mathcal{X}_{f,1} \cup \mathcal{X}_{f,2}\}$.

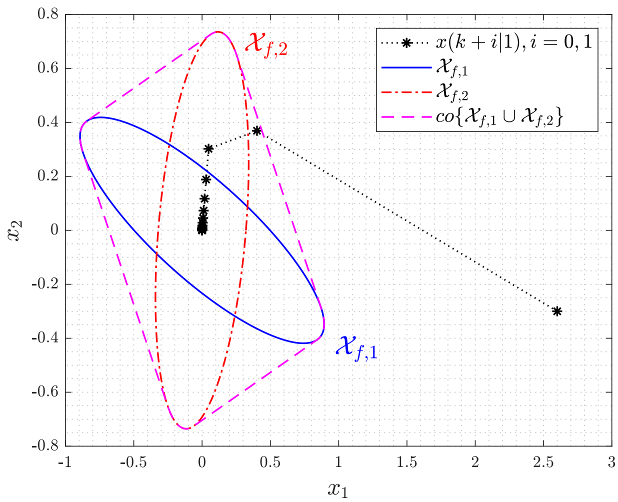


Fig. 7. State trajectory of the interpolation-based MPC when $N=1$ and $x_0=[2.6,\,0.3]$.

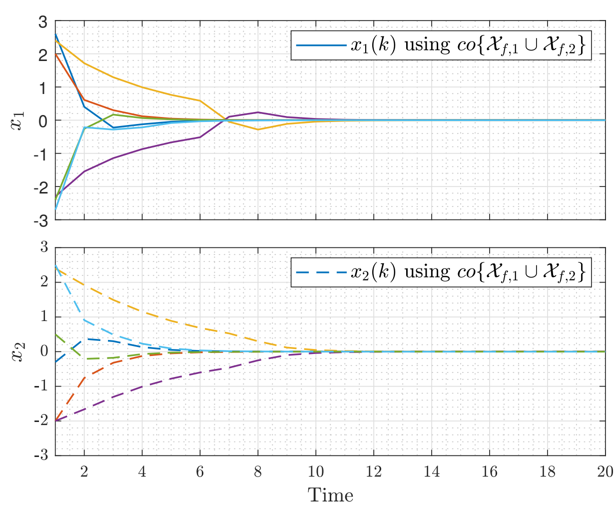


Fig. 8. State trajectories of the interpolation=based MPC for given different initial states.

5. CONCLUSION

The interpolation method on the MPC problem can enlarge the stability region of an existing MPC problem. This paper proposes a systematical approach to construct two invariant sets which can be used for interpolation-based MPC. The proposed scheme is systematic compared to the feedback-gain tuning algorithm. Numerical examples show that the MPC with interpolation has a larger stability region.

APPENDIX A

Since Lemma 6 holds true for K_{int}, $V(x)=x^\top X_{int}^{-1}x$, and $X_{f,int}=\{x\in\mathbb{R}^n|x^\top X_{int}^{-1}x\le\gamma\alpha_1+(1-\gamma)\alpha_2\}$ where $X_{int}=\gamma X_1+(1-\gamma)X_2$ for any $0\le\gamma\le 1$, for all $x\in\mathcal{X}_f$, we have

$$x^\top X_{int}^{-1}x-x^\top(A+BK_{int})^\top X_{int}^{-1}(A+BK_{int})x$$
$$\ge x^\top(Q+K_{int}^\top RK_{int})x. \tag{A.1}$$

Let $\gamma^*(k)$, and $u^*(k+i|k)$, $i=0,\ldots,N-1$, be the optimal solution of the problem at time k. For notational convenience, define

$$u_{i|k}^*=u^*(k+i|k),\ i=0,\ldots,N-1\,,$$
$$x_{i|k}^*=x^*(k+i|k),\ i=0,\ldots,N.$$

Thus, $x_{N|k}^*\in\mathcal{X}_{f,int}=\{x\in\mathbb{R}^n|x^\top X_{int}^{-1}(\gamma^*(k))x\le\gamma^*(k)\alpha_1+(1-\gamma^*(k))\alpha_2\}$ and the optimal vector $\mathcal{U}_k^*$ and $\mathcal{X}_k^*$ are given by

$$\mathcal{U}_k^*=\{u_{0|k}^*,u_{1|k}^*,\cdots,u_{N-1|k}^*\},$$
$$\mathcal{X}_k^*=\{x_{0|k}^*,x_{1|k}^*,\cdots,x_{N-1|k}^*,x_{N|k}^*\}.$$

Having this, the optimal cost function at time k is given by

$$J_N^*(\mathcal{U}_k^*,x(k))=\sum_{i=0}^{N-1}\ell(x_{i|k}^*,u_{i|k}^*)+x_{N|k}^{*\top}X_{int}^{-1}(\gamma^*(k))x_{N|k}^*. \tag{A.2}$$

Note that $\gamma^*(k)$ is a feasible γ at $k+1$. Moreover, the following is a feasible control sequence at $k+1$

$$\tilde{\mathcal{U}}_{k+1}=\{u_{1|k}^*,u_{2|k}^*,\cdots,u_{N-1|k}^*,K_{int}(x_{N|k}^*)\}. \tag{A.3}$$

The corresponding state prediction is given by

$$\tilde{\mathcal{X}}_{k+1}:=\{x_{1|k}^*,x_{2|k}^*,\cdots,x_{N-1|k}^*,x_{N|k}^*,\ (A+BK_{int})x_{N|k}^*\}, \tag{A.4}$$

where $(A+BK_{int})x_{N|k}^*\in\mathcal{X}_{f,int}$ due to the invariant $\mathcal{X}_{f,int}$ with K_{int}. Using (A.3) and (A.4), the cost function at $k+1$ is

$$\begin{aligned}&J_N(\tilde{\mathcal{U}}_{k+1},x_{k+1})\\&=\sum_{i=1}^{N-1}\ell(x_{i|k}^*,u_{i|k}^*)+\ell(x_{N|k}^*,K_{int}x_{N|k}^*)\\&\quad+x_{N|k}^{*\top}(A+BK_{int})^\top X_{int}^{-1}(\gamma^*(k))(A+BK_{int})x_{N|k}^*.\end{aligned} \tag{A.5}$$

Note that

$$\begin{aligned}&J_N(\tilde{\mathcal{U}}_{k+1},x_{k+1})\\&=J_N^*(\mathcal{U}_k^*,x(k))-\ell(x_{0|k}^*,u_{0|k}^*)-x_{N|k}^{*\top}X_{int}^{-1}(\gamma^*(k))x_{N|k}^*\\&\quad+\ell(x_{N|k}^*,K_{int}x_{N|k}^*)\\&\quad+x_{N|k}^{*\top}(A+BK_{int})^\top X_{int}^{-1}(\gamma^*(k))(A+BK_{int})x_{N|k}^*.\end{aligned} \tag{A.6}$$

Moreover, with (A.1) and $x_{N|k}^*\in\mathcal{X}_{f,int}$ in mind, we have

$$\begin{aligned}&x_{N|k}^{*\top}\left[(A+BK_{int})^\top X_{int}^{-1}(\gamma^*(k))(A+BK_{int})-X_{int}^{-1}\right](\gamma^*(k))\\&\quad\times x_{N|k}^*+\ell(x_{N|k}^*,K_{int}x_{N|k}^*)\le 0.\end{aligned}$$

Using this for (A.6) yields $J_N(\tilde{\mathcal{U}}_{k+1},x_{k+1})\le J_N^*(\mathcal{U}_k^*,x(k))$. Since $J_N^*(\mathcal{U}_{k+1}^*,x(k+1))\le J_N(\tilde{\mathcal{U}}_{k+1},x_{k+1})$, it follows that $J_N^*(\mathcal{U}_{k+1}^*,x(k+1))\le J_N^*(\mathcal{U}_k^*,x(k))$.

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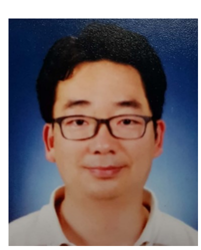

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